ON THE RELATION OF THE GENERALIZED ORTHOGONALITY OF P.F.PAPKOVICH FOR RECTANGULAR PLATES

(O SOOTNOSHENII OBOBSHCHENNOI ORTOGONAL'NOSTI P.F.PAPKOVICHA DLIA PRIAMOUGOL'NOI PLASTINKI)

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V.K. PROKOPOV
(Leningrad)

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1. The biharmonic equation in Cartesian coordinates $\Delta\Delta W = 0$ allows particular solutions of the form

$$W_k = e^{-\beta_k x} F_k(y) \tag{1.1}$$

applicable to problems on equilibrium of thin plates. In the case of bending, W is the deflection; for a plane problem, W is the stress function.

The functions $F_k(y)$ satisfy the differential equation

$$F_k^{IV} + 2\beta_k^2 F_k'' + \beta_k^4 F_k = 0 \tag{1.2}$$

and the parameters β_k are determined from the boundary conditions of the problem. For example, when conditions

$$F_k(\pm 1) = 0, \qquad F_k'(\pm 1) = 0 \tag{1.3}$$

are satisfied, which corresponds to the absence of stresses along the boundaries $y = \pm 1$ in the plane problem of the theory of elasticity, or to built-in supports of these edges in the bending problem, the parameters β_k will be the roots of the transcendental equation $\sin\beta \cos\beta \pm \beta = 0$.

For this case, the following result is due to Papkovich [1 and 2]; he found the relation of "generalized orthogonality"

$$\int_{-1}^{1} (F_k'' F_s'' - \beta_k^2 \beta_s^2 F_k F_s)\, dy = 0 \qquad (k \neq s) \tag{1.4}$$

which is satisfied by functions $F_k(y)$, when conditions (1.3) are present.

However, relation (1.4) exists not only when conditions (1.3) are satisfied. To prove this, we will reproduce the derivation of Equation (1.4) without using the requirements (1.3). Multiplying Equation (1.2) for number k by $\beta_s^2 F_s(y)$, and that for number s by $\beta_k^2 F_k(y)$, subtracting and integrating from -1 to $+1$, we get

$$\beta_s^2 \int_{-1}^{1} F_k^{IV} F_s dy - \beta_k^2 \int_{-1}^{1} F_k F_s^{IV} dy + 2\beta_k^2\beta_s^2 \int_{-1}^{1} (F_k'' F_s - F_k F_s'')\, dy +$$
$$+ \beta_k^2\beta_s^2 (\beta_k^2 - \beta_s^2) \int_{-1}^{1} F_k F_s dy = 0$$

Integration by parts of the first three integrals results in Expression

$$(\beta_s^2 - \beta_k^2) \int_{-1}^{1} (F_k'' F_s'' - \beta_k^2\beta_s^2 F_k F_s)\, dy + \tag{1.5}$$
$$+ [\beta_s^2 (F_k''' F_s - F_k'' F_s') - \beta_k^2 (F_k F_s''' - F_k' F_s'') + 2\beta_k^2\beta_s^2 (F_k' F_s - F_k F_s')]_{y=-1}^{y=+1} = 0$$

It is obvious, that when conditions (1.3) are satisfied, Equation (1.4) follows immediately from Expression (1.5). Consider now the case of the free edges of a plate under flexure; here we have

$$\frac{\partial^2 W_k}{\partial y^2} + \nu \frac{\partial^2 W_k}{\partial x^2} = 0, \qquad \frac{\partial^3 W_k}{\partial y^3} + (2 - \nu) \frac{\partial^3 W_k}{\partial x^2 \partial y} = 0 \qquad \text{for } y = \pm 1$$

or, from Equation (1.1),

$$F_k'' (\pm 1) + \nu\beta_k^2 F_k (\pm 1) = 0, \qquad F_k''' (\pm 1) + (2 - \nu)\, \beta_k^2 F_k' (\pm 1) = 0 \tag{1.6}$$

The substitution of Equations (1.6) into Expression (1.5) again leads to the relation (1.4).

For a thin plate, subjected to the conditions of a plane problem, with Airy's function expressible by Equation (1.1), we have the displacements

$$Eu_k = e^{-\beta_k x} \left[\nu\beta_k F_k (y) - \frac{F_k'' (y)}{\beta_k} \right], \qquad Ev_k = - e^{-\beta_k x} \left[(2 + \nu)\, F_k' (y) + \frac{F_k''' (y)}{\beta_k^2} \right]$$

Therefore, with the clamping of sides $y = \pm 1$ of such a plate, we have conditions

$$F_k'' (\pm 1) - \nu\beta_k^2 F_k (\pm 1) = 0, \qquad F_k''' (\pm 1) + (2 + \nu)\, \beta_k^2 F_k' (\pm 1) = 0 \tag{1.7}$$

Conditions (1.7) differ from (1.6) only in the sign of Poisson's ratio; relationship (1.4) will be also satisfied by conditions (1.7). It is also clear, that with the presence along the boundaries $y = +1$ and $y = -1$ of different conditions, belonging to one of the discussed cases (conditions (1.3) or (1.6) or (1.7)) *, the relation of the generalized orthogonality (1.4) will also take place. It must only be emphasized that the trancendental equation ** which determines the proper numbers β_k , as well as the form of the actual functions $F_k(y)$, depend essentially on the boundary conditions at $y = \pm 1$.

Grinberg [3] showed that Papkovich's relation can be expressed in different forms, for example

$$2 \int_{-1}^{1} F_k' F_s' dy - (\beta_k^2 + \beta_s^2) \int_{-1}^{1} F_k F_s dy = 0 \qquad (k \neq s) \tag{1.8}$$

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* Or conditions $F_k (\pm 1) = F_k'' (\pm 1) = 0$, which correspond to a supported edge of the plate.

** The transcendental equation for the proper numbers β_k and the forms of the proper functions $F_k(y)$, for various boundary conditions of a rectangular plate can be found in the article by Kitover [4]. See also the book by Ufliand [5] which gives the values of the first numbers β_k .

It is easy to check, that the relationship (1.8) will also take place in the above mentioned cases.

2. Relation (1.4) was used by Papkovich to satisfy the boundary conditions in the problem of plate bending with rigidly built-in edges $y = \pm 1$ [2]. For any boundary conditions at $x = 0$, the problem of determining the coefficients a_k in the homogeneous solution

$$W = \sum_k a_k W_k = \sum_k a_k e^{-\beta_k x} F_k(y) \tag{2.1}$$

by the method of Papkovich, reduces to some integral equation [2 and 3]. However, for the case of a supported edge $x = 0$, or an edge with a roller, it was shown by Papkovich, that relation (1.4) enables us to obtain a general formula for the separate determination of the coefficients a_k. Since the homogeneous solution is added to the particular integral, corresponding to the loading (and the boundary conditions at $y = \pm 1$), at a supported edge $x = 0$ we can consider as given the deflection and bending moment

$$W|_{x=0} = \varphi_1(y), \qquad \frac{\partial^2 W}{\partial x^2} + \nu \frac{\partial^2 W}{\partial y^2}\bigg|_{x=0} = \varphi_2(y)$$

Substituting W from (2.1), we get

$$\sum_k a_k F_k(y) = \varphi_1(y), \qquad \sum_k a_k \left[\beta_k^2 F_k(y) + \nu F_k''(y)\right] = \varphi_2(y) \tag{2.2}$$

Eliminating $F_k''(y)$ from the second equation (2.2), by means of the first equation, we get a simpler system of equations

$$\sum_k a_k F_k(y) = \varphi_1(y) = f_1(y), \qquad \sum_k a_k \beta_k^2 F_k(y) = \varphi_2(y) - \nu\varphi_1''(y) = f_2(y) \tag{2.3}$$

Papkovich gives the following solution to the problem of determining the coefficients a_k from conditions (2.3), we form the difference

$$F_s''(y) f_1''(y) - \beta_s^2 F_s(y) f_2(y) = \sum_k a_k \left(F_s'' F_k'' - \beta_s^2 \beta_k^2 F_s F_k\right)$$

integrating this from -1 to $+1$ and using (1.4), we get

$$a_s = \frac{1}{I_s} \int_{-1}^{+1} \left\{F_s''(y) f_1''(y) - \beta_s^2 F_s(y) f_2(y)\right\} dy \tag{2.4}$$

where

$$I_s = \int_{-1}^{+1} \left\{[F_s''(y)]^2 - \beta_s^4 [F_s(y)]^2\right\} dy \tag{2.5}$$

With a roller support of edge $x = 0$, the angle of rotation and the support reaction are given by

$$\frac{\partial W}{\partial x}\bigg|_{x=0} = \varphi_1(y), \qquad \frac{\partial^3 W}{\partial x^3} + (2-\nu)\frac{\partial^3 W}{\partial x \partial y^2}\bigg|_{x=0} = \varphi_2(y)$$

The substitution into these conditions of the series (2.1) leads to equations

$$-\sum_k a_k \beta_k F_k(y) = \varphi_1(y), \qquad -\sum_k a_k \beta_k \left[\beta_k^2 F_k(y) + (2-\nu) F_k''(y)\right] = \varphi_2(y) \tag{2.6}$$

It is easy to see that by substituting the coefficients $-a_k\beta_k = b_k$ and by introducing functions

$$f_1(y) = \varphi_1(y), \qquad f_2(y) = \varphi_2(y) - (2-\nu)\varphi_1''(y)$$

Equations (2.6) are brought to a form, identical to the system (2.3)

$$\sum_k b_k F_k(y) = f_1(y), \qquad \sum_k b_k \beta_k^2 F_k(y) = f_2(y)$$

In the case of a plane problem for a rectangular strip, the following combinations can be solved. One, in which at the edge $x = 0$ we are given the normal displacement u and the tangential stress τ_{xy}, and the other, when we are given the normal stress σ_x and the tangential displacement v. For the first case we have the conditions

$$Eu\,|_{x=0} = \sum_k a_k \left\{\nu\beta_k F_k(y) - \frac{F_k''(y)}{\beta_k}\right\} = \varphi_1(y), \qquad -\frac{\partial^2 W}{\partial x \partial y}\bigg|_{x=0} = \sum_k a_k \beta_k F_k'(y) = \varphi_2(y) \tag{2.7}$$

Let us assume, that at least one of the longitudinal edges of the strip is free from stress; let it be the side $y = -1$, then $F_k(-1) = 0$. We will introduce the functions

$$f_1(y) = \nu f_2(y) - \varphi_1(y), \qquad f_2(y) = \int_{-1}^{y} \varphi_2(y)\, dy \tag{2.8}$$

and the coefficients

$$b_k = a_k / \beta_k$$

Integrating the second condition (2.7) from -1 to y, and performing some simple transformations we get the system

$$\sum_k b_k F_k''(y) = f_1(y), \qquad \sum_k b_k \beta_k^2 F_k(y) = f_2(y) \tag{2.9}$$

which is even simpler than the system (2.3). Using the relation of "generalized orthogonality" (1.4), we obtain from (2.9) the following equation for the coefficients

$$b_k = \frac{1}{I_k} \int_{-1}^{1} \{F_k''(y)\, f_1(y) - \beta_k^2 F_k(y)\, f_2(y)\}\, dy \tag{2.10}$$

If both the longitudinal edges $y = \pm 1$ are rigidly built-in, then the transformation of the system (2.7) will be somewhat more complicated. Let us differentiate twice the first equation (2.7), using the differential equation (1.2) we get

$$\varphi_1''(y) = \sum_k a_k \{(2 + \nu)\, \beta_k F_k''(y) + \beta_k^3 F_k(y)\} \tag{2.11}$$

Let us now introduce the functions

$$\varphi_1''(y) - (2 + \nu)\, \varphi_2'(y) = f_2(y), \qquad \varphi_2'(y) = f_1(y) \tag{2.12}$$

and the coefficients

$$b_k = a_k \beta_k$$

for which we again get the system (2.9). We could have introduced the functions (2.12) also in the preceeding case, however their introduction requires an additional condition of the existence of the derivatives (φ_1'' and φ_2') on the right-hand side of conditions (2.7).

When, at the edge $x = 0$, we are given the normal stress and the tangential displacement, we have the conditions

$$\begin{aligned} \frac{\partial W^2}{\partial y^2}\bigg|_{x=0} &= \sum_k a_k F_k''(y) = \varphi_1(y) \\ Ev\,|_{x=0} &= -\sum_k a_k \left\{(2 + \nu)\, F_k'(y) + \frac{F_k'''(y)}{\beta_k^2}\right\} = \varphi_2(y) \end{aligned} \tag{2.13}$$

Differentiating the second condition (2.13) and using the differential equation (1.2), we get

$$\sum_k a_k \{\beta_k^2 F_k(y) - \nu F_k''(y)\} = \varphi_2'(y) \tag{2.14}$$

and after further introducing the functions

$$f_1(y) = \varphi_1(y), \qquad f_2(y) = \varphi_2'(y) + \nu\varphi_1(y) \tag{2.15}$$

we get equations, identical with the system (2.9)

$$\sum_k a_k F_k''(y) = f_1(y), \qquad \sum_k a_k \beta_k^2 F_k(y) = f_2(y) \tag{2.16}$$

Thus, it is possible to generalize the method of Papkovich to all the cases, when relation (1.4) is satisfied, which, as was shown in Section 1, is not necessarily connected with the conditions $F_k(\pm 1) = F_k'(\pm 1) = 0$.

3. In the solution of the plane problem of the theory of elasticity for a semi-infinite strip $x \geqslant 0$, the edges $y = \pm 1$ of which are free from external forces, the question arises, as to what are the conditions to be satisfied by the functions $\varphi_1(y)$ and $\varphi_2(y)$, which characterize the displacements along the edge $x = 0$, in order that the stresses (and displacements) should die away with distance from the edge. If, along the edge of the strip, we are given the stresses

$$\sigma_x|_{x=0} = \varphi_1(y), \qquad \tau_{xy}|_{x=0} = \varphi_2(y) \tag{3.1}$$

then, according to Saint-Venant's principle, the necessary condition for damping is that the resultant vector and moment of the forces (2.1) applied to the edge $x = 0$, should be equal to zero, i.e.

$$\int_{-1}^{1} \varphi_1(y)\,dy = 0, \qquad \int_{-1}^{1} \varphi_1(y)\,y\,dy = 0, \qquad \int_{-1}^{1} \varphi_2(y)\,dy = 0 \tag{3.2}$$

In other cases, the question needs a special investigation. In the case when either the normal displacement u and the tangential stress τ_{xy} (2.7), or the tangential displacement v and the normal stress σ_x (2.13) are given, such conditions are easily established. Since in the above mentioned cases the actual solution of the problem is determined by the systems (2.9) and (2.16) which are identical, then the conditions imposed on the functions $f_1(y)$ and $f_2(y)$ will be the same in both cases.

Let us investigate the problem (2.7). One of the conditions can be written immediately: it is the same as the last of conditions (3.2), since the function $\varphi_2(y)$ represents the edge value of the tangential stress τ_{xy}. In problem (2.13), the function $f_1(y)$ will be the given stress σ_x at $x = 0$, therefore also in problem (2.7) we must satisfy conditions

$$\int_{-1}^{1} f_1(y)\,dy = 0, \qquad \int_{-1}^{1} f_1(y)\,y\,dy = 0 \tag{3.3}$$

However, in problem (2.7) the function $f_1(y)$ is determined by Equations (2.8), therefore in place of (3.3) we will have

$$\nu\int_{-1}^{1} dy \int_{-1}^{y} \varphi_2(\eta)\,d\eta - \int_{-1}^{1} \varphi_1(y)\,dy = 0, \qquad \nu\int_{-1}^{1} y\,dy \int_{-1}^{y} \varphi_2(\eta)\,d\eta - \int_{-1}^{1} y\varphi_1(y)\,dy = 0$$

Changing the order of integration and using the last of conditions (3.2), we get the final expression of the additional conditions

$$\int_{-1}^{1} \{\varphi_1(y) + \nu y\varphi_2(y)\}\,dy = 0, \qquad \int_{-1}^{1} \left\{y\varphi_1(y) + \frac{\nu}{2} y^2\varphi_2(y)\right\} dy = 0 \tag{3.4}$$

The last of conditions (3.2) and conditions (3.4) must be applied to the functions $\varphi_1(y)$ and $\varphi_2(y)$ in the edge problem (2.7) for the semi-infinite strip, along the sides $y = \pm 1$ which have no stresses. By analogous considerations we reach the conditions

$$\int_{-1}^{1} \varphi_1(y)\, dy = 0, \quad \int_{-1}^{1} \varphi_1(y)\, y dy = 0, \quad \varphi_2'(1) + \nu\varphi_1(1) = \varphi_2'(-1) + \nu\varphi_1(-1) \qquad (3.5)$$

which take place in the edge problem (2.13) for the semi-infinite strip with stress-free edges $y = \pm 1$.

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